Course code: MA-431

Course name: MATHEMATICS FOR MECHATRONICS

Date: December 6, 2019

Duration: 5 hours, 9:00-14:00

Number of pages including the title page: 17

Resources allowed: Booklet, Calculators

Notes:

1. Consider the following second-order differential equation

$$y'' - y' - 2y = 4t^2.$$

   (a) Find the general solution of the corresponding homogeneous equation, demonstrate that your solution is a linear combination of two linearly independent solutions.

   (b) Find a particular solution of the non-homogeneous equation using the method of undetermined coefficients.

   (c) Find a particular solution of the non-homogeneous equation using the method of variation of parameters.

   (d) Write the general solution of the non-homogeneous equation.

   (e) Find the solution of the initial-value problem with the following initial conditions

$$y(0) = -2, \quad y'(0) = 3.$$

   (f) Answer the question: Is this solution unique? Explain why.

   Remark. In this problem, it is allowed to use calculators for finding derivatives, coefficients in Part b) and integrals in Part c).

2. A circuit has in series an electromotive force given by $E = 200e^{-100t}\ V$, a resistor of 10 Ω, an inductor of 0.05 H and a capacitor of $2 \times 10^{-4}\ F$. If the initial current and the initial charge on the capacitor are both zero, find the charge on the capacitor $q(t)$ at any time $t > 0$. Explain the behavior of the solution when $t \to \infty$.

   Remark. Use of calculators for finding coefficients is accepted.

3. Use the Laplace transform to find the solution of the following initial value problem

$$\begin{aligned} y'' + y &= u_{3\pi}(t), \\ y(0) &= 1, \\ y'(0) &= 0. \end{aligned}$$

   Remark. Use of calculators for finding coefficients is accepted.

4. Consider the system

$$\frac{d\overrightarrow{x}}{dt}=\begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}\overrightarrow{x}.$$

   (a) Find the eigenvalues and eigenvectors.

   (b) Find the general solution.

   (c) Classify the critical point $(0,0)$ according to its type and stability properties.

   (d) Explain the behavior of the solutions as t increases infinitely.

   (e) Sketch the phase portrait of the system.

   (f) Solve the nonhomogeneous system

$$\frac{d\overrightarrow{x}}{dt}=\begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}\overrightarrow{x}+\begin{pmatrix}6e^{t}\\ -6e^{2t}\end{pmatrix}.$$

5. Given the autonomous system

$$\begin{aligned}\frac{dx}{dt} &= x\,(y-1)\,,\\ \frac{dy}{dt} &= y\,(2-x-y)\,.\end{aligned}$$

   (a) Determine all critical points of the given system of equations.

   (b) Find the corresponding linear system near each critical point.

   (c) Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?

6. Consider the system

$$\begin{aligned}x' &= -y-x^5,\\ y' &= x-y^5.\end{aligned}$$

   (a) Show that the system is locally linear near the equilibrium point $(0,0)$.

   (b) Find the corresponding linear system.

   (c) Classify the critical point $(0,0)$ for the linear system according to the type and stability.

(d) What can you say in this case about the behavior of the nonlinear system?

(e) Prove the stability of the critical point $(0,0)$ for the nonlinear system using an appropriate Liapunov function.

7. Consider the following nonlinear system

$$\begin{aligned} x' &= y + x\left(25 - x^2 - y^2\right), \\ y' &= -x + y\left(25 - x^2 - y^2\right). \end{aligned}$$

(a) Transform the system to polar coordinates.

(b) Find all periodic solutions of the system and determine their stability.

GOOD LUCK!

Integration Formulas

1. $\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$	2. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$
3. $\int \cos x dx = \sin x + C$	4. $\int \sin x dx = -\cos x + C$
5. $\int \tan x dx = -\ln\lvert\cos x\rvert + C$	6. $\int \cot x dx = \ln\lvert\sin x\rvert + C$
7. $\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$	8. $\int \csc x dx = -\ln\lvert\csc x + \cot x\rvert + C$
9. $\int \sec x \tan x dx = \sec x + C$	10. $\int \csc x \cot x dx = -\csc x + C$
11. $\int \sec^2 x dx = \tan x + C$	12. $\int \csc^2 x dx = -\cot x + C$
13. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$	14. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$
15. $\int \frac{dx}{\lvert x\rvert\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{x}{a} + C$	16. $\int e^x dx = e^x + C$
17. $\int \frac{x^2 dx}{1+x^2} = x - \arctan x + C$	

$$\begin{aligned}
\int e^{ax}\sin bx dx &= \frac{1}{a^2+b^2}e^{ax}\left[a\sin bx - b\cos bx\right] \\
\int e^{ax}\cos bx dx &= \frac{1}{a^2+b^2}e^{ax}\left[a\cos bx + b\sin bx\right]
\end{aligned}$$

Integration and differentiation rules

$$\begin{aligned}
(f+g)' &= f'+g', \\
(fg)' &= f'g+fg', \\
\left(\frac{f}{g}\right)' &= \frac{f'g-fg'}{g^2}, \\
(f(g(x)))' &= f'(g(x))\,g'(x), \\
\int (f+g)\,dx &= \int f dx + \int g dx, \\
\int u dv &= uv - \int v du, \\
\int f(g(x))\,d(g(x)) &= \int f(u)\,du.
\end{aligned}$$

Some Useful Formulas

$$
\begin{aligned}
e^{(a\pm ib)t} &= e^{at}\left(\cos bt \pm i\sin bt\right), \\
s^2+as+b &= \left(s+\frac{a}{2}\right)^2 + b - \frac{a^2}{4}, \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ..., \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ..., \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} ... + (-1)^n \frac{x^{2n}}{(2n)!} + ...
\end{aligned}
$$

$$
\begin{aligned}
\cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right), \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \\
\cos^2 x &= \frac{1}{2}\left(1+\cos 2x\right), \qquad \sin^2 x = \frac{1}{2}\left(1-\cos 2x\right), \\
\csc x &= \frac{1}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \\
\sin\left(\frac{\pi}{2}-x\right) &= \cos x, \qquad \cos\left(\frac{\pi}{2}-x\right) = \sin x, \\
\sin(-x) &= -\sin x, \qquad \cos(-x) = \cos x, \\
\sin 2x &= 2\sin x\cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x, \\
\sin x \sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right], \\
\cos x \cos y &= \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right], \\
\sin x \cos y &= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right].
\end{aligned}
$$

Partial fractions

$$\begin{aligned} As^2 + Bs + C &= 0 \quad \text{has the roots:} \\ r_{1,2} &= \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \qquad \text{then} \end{aligned}$$

$$\frac{as+b}{As^2+Bs+C} = \begin{cases} \frac{as+b}{A(s-r_1)(s-r_2)}, & \text{if } r_1, r_2 \text{ are real,} \\ \frac{as+b}{A\left[(s-\alpha)^2+\beta^2\right]}, & \text{if } r_{1,2} = \alpha \pm i\beta \text{ - complex conjugate.} \end{cases}$$

Separable equations

$$\begin{aligned} \frac{dy}{dx} &= f_1(x) f_2(y), \\ M(x) + N(y)\frac{dy}{dx} &= 0, \\ M(x)dx + N(y)dy &= 0. \end{aligned}$$

Exact equations

$$\begin{aligned} M(x,y)dx + N(x,y)dy &= 0, \\ \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x}, \\ \psi(x,y) &= \int M(x,y)dx + \int \left[N(x,y) - \int M_y(x,y)dx \right] dy. \end{aligned}$$

Classes of UC functions

$$\begin{aligned} 1) \quad P_n(t) &= a_0 t^n + a_1 t^{n-1} + ... + a_{n-1} t + a_n, \\ 2) \quad P_n(t)e^{\alpha t} &= \left(a_0 t^n + a_1 t^{n-1} + ... + a_{n-1} t + a_n \right) e^{\alpha t}, \\ 3) \quad P_n(t)e^{\alpha t}\cos\beta t &= \left(a_0 t^n + a_1 t^{n-1} + ... + a_{n-1} t + a_n \right) e^{\alpha t}\cos\beta t, \\ P_n(t)e^{\alpha t}\sin\beta t &= \left(a_0 t^n + a_1 t^{n-1} + ... + a_{n-1} t + a_n \right) e^{\alpha t}\sin\beta t, \end{aligned}$$

where n is a nonnegative integer, α and β are real numbers.

A particular solution of

$$ay'' + by' + cy = g(t)$$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + ... + a_{n-1}t + a_n$	$t^s\left(A_0t^n + A_1t^{n-1} + ... + A_{n-1}t + A_n\right)$
$P_n(t)e^{\alpha t}$	$t^s\left(A_0t^n + A_1t^{n-1} + ... + A_{n-1}t + A_n\right)e^{\alpha t}$
$P_n(t)e^{\alpha t}\cos\beta t, \quad P_n(t)e^{\alpha t}\sin\beta t$	$t^s\left[Q_{n1}(t)e^{\alpha t}\cos\beta t + Q_{n2}(t)e^{\alpha t}\sin\beta t\right]$

where

$$\begin{aligned} Q_{n1}(t) &= A_0t^n + A_1t^{n-1} + ... + A_{n-1}t + A_n, \\ Q_{n2}(t) &= B_0t^n + B_1t^{n-1} + ... + B_{n-1}t + B_n, \end{aligned}$$

and s is the smallest nonnegative integer ($s = 0, 1$, or 2), that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation.

If r_1 and r_2 are the roots of the characteristic equation then the general solution of the homogeneous DE can be found in the following form

Roots r_1 and r_2	General solution
r_1 and r_2 are real and unequal	$y(t) = C_1e^{r_1t} + C_2e^{r_2t}$
r_1 and r_2 are real and equal, $r_1 = r_2 = r$	$y(t) = C_1e^{rt} + C_2te^{rt}$
r_1 and r_2 are complex conjugate, $r_{1,2} = \alpha \pm i\beta$	$y(t) = C_1e^{\alpha t}\cos\beta t + C_2e^{\alpha t}\sin\beta t$

A particular solution of the nonhomogeneous equation can be found as

$$g(t) = e^{at} \qquad y_p(t) = \begin{cases} Ae^{at}, & \text{if } r_1 \neq a,\ r_2 \neq a \\ Ate^{at}, & \text{if } r_1 = a,\ r_2 \neq a \\ At^2e^{at}, & \text{if } r_1 = r_2 = a \end{cases}$$

$$g(t) = K_1 \cos(\beta t) + K_2 \sin(\beta t) \quad y_p(t) = \begin{cases} A_1 \cos(\beta t) + A_2 \sin(\beta t), & \text{if } r \neq \pm\beta i \\ t\left(A_1 \cos(\beta t) + A_2 \sin(\beta t)\right) & \text{if } r = \pm\beta i \end{cases}$$

$$g(t) = P_n(t) \qquad y_p(t) = \begin{cases} Q_n(t) & \text{if } r_1 \neq 0,\ r_2 \neq 0 \\ tQ_n(t) & \text{if } r_1 = 0,\ r_2 \neq 0 \\ t^2Q_n(t) & \text{if } r_1 = r_2 = 0 \end{cases}$$

where

$$\begin{aligned} P_n(t) &= a_0t^n + a_1t^{n-1} + ... + a_{n-1}t + a_n, \\ Q_n(t) &= A_0t^n + A_1t^{n-1} + ... + A_{n-1}t + A_n. \end{aligned}$$

Charge in Electrical Circuit

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = 0,$$

or

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0,$$

Damped Free Vibration

$$mu'' + \gamma u' + ku = 0$$

Variation of Parameters

For DE

$$ay'' + by' + cy = g(t)$$

the particular solution is found in the form

$$Y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

where $y_1(t), y_2(t)$ is the fundamental set of the corresponding homogeneous equation, $v_1(t), v_2(t)$ are unknown functions their derivatives satisfy the following system

$$\begin{aligned} v_1'(t)y_1(t) + v_2'(t)y_2(t) &= 0, \\ v_1'(t)y_1'(t) + v_2'(t)y_2'(t) &= \frac{g(t)}{a}. \end{aligned}$$

If the functions $p(t)$, $q(t)$, and $g(t)$ are continuous on the open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous equation

$$ay'' + by' + cy = 0,$$

then a particular solution of the nonhomogeneous equation

$$ay'' + by' + cy = g(t)$$

is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds$$

where t_0 is any conveniently chosen point in I. The general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t).$$

Laplace Transform

$$L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

Laplace Transform Table

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, \quad n$ is a positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, \quad n$ is a positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s), \quad F(s) = L\{f(t)\}$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$

Inverse Matrix Formula

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Linear Systems with Constant Coefficients

$$x' = Ax.$$

The solutions has the form

$$x = \xi e^{rt}, \text{where } r \text{ is the eigenvalue, } \xi \text{ is the eigenvector}$$

Equation for Eigenvalues

$$\det (A - rI) = 0.$$

Equation for Eigenvectors

$$(A - rI)\,\xi = 0.$$

In case $r_1 = r_2 = r$ and there is only one eigenvector corresponding to r, the form of the second solution is

$$x = \xi t e^{rt} + \eta e^{rt},$$

where η satisfies

$$(A - rI)\,\eta = \xi.$$

Nonhomogeneous Linear System

$$x' = P(t)x + g(t),$$

has the solution

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x^0 + \Phi(t)\int_0^t \Phi^{-1}(s)g(s)ds$$

where $\Phi(t)$ is the fundamental matrix of the corresponding homogeneous system.

Stability Properties of Linear and Almost Linear Systems

The nonlinear system $x' = Ax + g(x)$ called locally linear about the equilibrium point $x = 0$ if

$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

is such that

$$\frac{\|g(x)\|}{\|x\|} \to 0, \quad \text{as} \quad x \to 0,$$

or

$$\frac{g_1(x,y)}{r} \to 0, \qquad \frac{g_2(x,y)}{r} \to 0, \qquad \text{as} \qquad r \to 0,$$

where

$$r = \|g(x)\| = \left[g_1^2(x,y) + g_2^2(x,y)\right]^{1/2}.$$

	Linear System		Locally Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	U	N	U
$r_1 < r_2 < 0$	N	AS	N	AS
$r_1 < 0 < r_2$	SP	U	SP	U
$r_1 = r_2 > 0$	PN or IN	U	N or SpP	U
$r_1 = r_2 < 0$	PN or IN	AS	N or SpP	AS
$r_1, r_2 = \lambda \pm i\mu, \lambda > 0$	SpP	U	SpP	U
$r_1, r_2 = \lambda \pm i\mu, \lambda < 0$	SpP	AS	SpP	AS
$r_1 = i\mu, r_2 = -i\mu$	C	S	C or SpP	I

Notation: N-node, PN - proper node, IN-improper node, SP - saddle point, SpP - spiral point, C - center, U - unstable, AS-asymptotically stable, S - stable, I-indeterminate.

Jacobian for Autonomous System

$$\begin{aligned} \frac{dx}{dt} &= F(x,y), \\ \frac{dy}{dt} &= G(x,y), \end{aligned}$$

is

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}.$$

Lyapunov's Theorems

Theorem 1 *Suppose that an autonomous system*

$$\begin{aligned}\frac{dx}{dt} &= F(x,y), \\ \frac{dy}{dt} &= G(x,y),\end{aligned}$$

has an isolated critical point at the origin. If there exists a function V that is continuous and has continuous first partial derivatives, that is positive definite and for which the function

$$\dot{V}(x,y) = V_x(x,y)\,F(x,y) + V_y(x,y)\,G(x,y)$$

is negative definite for some domain D in the $xy-$ plane containing the point $(0,0)$, then the origin is an asymptotically stable critical point. If $\dot{V}$ is negative semidefinite then the origin is a stable critical point.

Theorem 2 *Suppose that an autonomous system*

$$\begin{aligned}\frac{dx}{dt} &= F(x,y), \\ \frac{dy}{dt} &= G(x,y)\end{aligned}$$

has an isolated critical point at the origin. Let V be a function that is continuous and has continuous first partial derivatives. Suppose that $V(0,0)=0$ and in every neighborhood of the origin there is at least one point at which V is positive (negative). If there exists a domain D containing the origin such that the function

$$\dot{V}(x,y) = V_x(x,y)\,F(x,y) + V_y(x,y)\,G(x,y)$$

is positive definite (negative definite) on D, then the origin is an unstable critical point.

Theorem 3 *Suppose that an autonomous system*

$$\begin{aligned}\frac{dx}{dt} &= F(x,y), \\ \frac{dy}{dt} &= G(x,y)\end{aligned}$$

has an isolated critical point at the origin. Let V be a function that is continuous and has continuous first partial derivatives. If there is a bounded domain D_K containing the origin where $V(x,y) < K$ for some positive K, V is positive definite and $\dot{V}$ is negative definite, then every solution of the system that starts at a point in D_K approaches the origin as t approaches infinity.

Theorem 4 *The function*

$$V(x,y) = ax^2 + bxy + cy^2$$

is positive definite if and only if

$$a > 0, \qquad 4ac - b^2 > 0,$$

and is negative definite if and only if

$$a < 0, \qquad 4ac - b^2 > 0.$$

Limit Cycles of the System

$$\begin{aligned} \frac{dx}{dt} &= F(x,y), \\ \frac{dy}{dt} &= G(x,y). \end{aligned}$$

Theorem 5 *Let the functions F and G have continuous first partial derivatives in the domain D of the xy-plane. A closed trajectory of the system must necessarily enclose at least one critical point, the critical point can not be a saddle point.*

Theorem 6 *Let the functions F and G have continuous first partial derivatives in the simply connected domain D of the xy-plane. If*

$$F_x + G_y$$

have the same sign throughout D then there is no closed trajectory of the system lying entirely in D. (A simply connected two-dimensional domain is the domain with no holes).

Theorem 7 *Let the functions F and G have continuous first partial derivatives in the domain D of the $xy-$ plane. Let D_1 be a subdomain of D, and let R be a region that consists of D_1 and its boundary (all points of R are in D). Suppose that R contains no critical points of the system. If there exists a constant t_0 such that*

$$\begin{aligned} x &= \phi(t), \\ y &= \psi(t) \end{aligned}$$

is a solution that exists and stays in R for all $t \geq t_0$, then either

$$\begin{aligned} x &= \phi(t), \\ y &= \psi(t) \end{aligned}$$

is a periodic solution, or it spirals toward a closed trajectory as $t \to \infty$. In either case the system has a periodic solution in R.

Existence and Uniqueness Theorems

Theorem 8 (Theorem 2.4.1 (page 69)) *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \varphi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Theorem 9 (Theorem 2.4.2 (page 70)) *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \varphi(t)$ of the initial value problem*

$$y = f(t, y), \qquad y(t_0) = y_0.$$

Theorem 10 (Theorem 2.8.1 (page 113)) *If f and $\partial f/\partial y$ are continuous in a rectangle $R : |t| \le a,\ |y| \le b$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution $y = \varphi(t)$ of the initial value problem*

$$y' = f\left(t, y\right), \qquad y\left(0\right) = 0.$$

Theorem 11 (Theorem 3.2.1 (page 146) (Existence and Uniqueness Theorem)) *Consider the initial value problem*

$$\begin{aligned} y'' + p(t)y' + q(t)y &= g(t), \\ y(t_0) &= y_0, \\ y'(t_0) &= y_0^1, \end{aligned}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists and twice continuously differentiable through the interval I.