Eksamen MA-433, høst 2019

Emnekode:	MA-433
Emnenavn:	Partielle differensiallikninger og harmonisk analyse
Dato:	2 des 2019
Varighet:	5 timer
Antall ark:	1
Tillatte hjelpemidler:	Ingen

Skriv studentnummer på alle innleverte ark.
Skriv bare på **en side per ark.**
Skriv **høyst en oppgave per ark.**
Alle svar MÅ begrunnes for å gi poeng.

- **OPPGAVE 1:** Let $f(x) = |x|$ on $[-\pi, \pi]$ be extended to a $2\pi-$periodic function.

  a) Show that the Fourier series for f is

  $$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$$

  b) Give arguments that show

  $$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}, \text{ whenever } -\pi \le x \le \pi$$

  Also, compute the sum $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

  c) Show that

  $$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

  d) Clarify why, and in what sense, the following equality holds on $[-\pi, \pi]$:

  $$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

- **OPPGAVE 2:** Let g_k, $k = 1, 2, 3, \ldots$, be a family of good kernels on $\mathbb{R}$ and suppose that $f : \mathbb{R} \to \mathbb{C}$ is bounded and continuous. Show that

$$\text{for every } x \in \mathbb{R}: \qquad \lim_{k\to\infty} (g_k * f)(x) = f(x)$$

- **OPPGAVE 3:** Let the function f be defined by

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ \pi & x = 0 \\ -e^{x} & x < 0 \end{cases}$$

a) Show that the Fourier transform for f is

$$\hat{f}(\xi) = \frac{1}{1 + \mathrm{i}\xi} - \frac{1}{1 - \mathrm{i}\xi}$$

b) Calculate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(1 + x^2)^2}\, dx$$

c) Does the inversion formula hold for f at $x = 0$?

- **OPPGAVE 4:** Consider the fourth order Schrödinger equation

$$\begin{cases} u_t = \mathrm{i}\, u_{xxxx}, & t > 0, x \in \mathbb{R} \\ u(0, x) = f(x), & x \in \mathbb{R} \end{cases}$$

a) Find a solution candidate $u(t, x)$ using the Fourier transform and show that

$$f \in L^2(\mathbb{R}) \implies \lim_{t\to 0} u(t) = f \text{ in } L^2(\mathbb{R})$$

b) Show that

$$f,\ \xi^4 \widehat{f} \in L^2(\mathbb{R}) \implies \frac{\partial u}{\partial t} \text{ exists in } L^2(\mathbb{R})$$