Archimedes (287 – 212 BC)	
Euclid X.1 ($B/2^n < \varepsilon$)	Archimedean postulate ($B/n < \varepsilon$)
Quadrature of Parabola (by Arch) using sum of finite geometric progression	**Area of spiral (by Arch)** using $\Sigma i^2 = (n/6)(n + 1)(2n + 1)$
1593 Viète: sum of infinite GP (wobble), infinite product for π	1636 Fermat: area under $y = x^n$
1647 Gregory of St Vincent: sum of infinite GP as a limit	1643 Fermat used sum of infinite GP for area under $y^p = x^q$ and $x^p y^q = 1$
1655 Wallis: infinite product for π - "as small as one might wish"	
1657 Fermat found areas of particular curves as limits	
1687 Newton shows area under monotonic graph as limit	

THE VICE
If $-\varepsilon < A < \varepsilon$ for all positive ε, then $A = 0$.
If $-\varepsilon < A - K < \varepsilon$ for all positive ε, then $A = K$.

Euclid X.1 gives
if $-(½)^n < A < (½)^n$ for all positive integers n, then $A = 0$.

Archimedean postulate gives
if $-1/n < A < 1/n$ for all positive integers n, then $A = 0$.

Generally
if $-a_n < A < a_n$ for all positive integers n, and the sequence (a_n) contains arbitrarily small terms, then $A = 0$.
Small terms are far along the sequence.
If (b_n) is a sequence of positive terms and contains arbitrarily small terms, then $-b_n < B < b_n$, for all n implies $B = 0$.
But without further restriction, $-(a_n + b_n) < A + B < a_n + b_n$, for all n, does not imply that $A + B$ is small, let alone 0!
To make the algebra work, and force $A + B = 0$, small terms in (a_n) and (b_n) must overlap.
This can be guaranteed by insisting that (a_n) is a sequence of positive terms, such that for large enough n, a_n is as small as we like. [Given $\varepsilon > 0$, $n > N$ makes $a_n < \varepsilon$.] Call such a sequence a null sequence of positive terms.

Define **limit** L of sequence (A_n), where $-a_n < A_n - L < a_n$, for all n, where (a_n) is a null sequence of positive terms.

Wallis (1655) obtained inequalities of the form $A_n < 4/\pi < B_n$, where (A_n) is increasing, (B_n) is decreasing and $(B_n - A_n)$ is a null sequence of positive terms. Thus $0 < 4/\pi - A_n < B_n - A_n$. So (A_n) has limit $4/\pi$.

Newton (1687) obtained inequalities of the form $I_n < A < C_n$, where A is the area under a monotonic graph, I_n the sum of inscribed rectangles, C_n the sum of circumscribed rectangles. Since $C_n - I_n = k/n$, the argument as for Wallis shows that the sequence (I_n) has limit A.

There was no attempt to justify the algebra of limits before the end of the 18th century.

Exercise

The method of Archimedes in determining that the area of a spiral is one third the area of its circumscribing circle may be echoed to prove the following results **using the vice but not using limits.**

1. (Fermat, 1636) Find the area of n inscribed and n circumscribed rectangles of equal width to show that the area bounded by the parabola $y = x^2$, the x-axis and $x = a$ is $(1/3)a^3$.

2. (Originally Eudoxos, reported in Euclid XII.5-7, but there, much harder than Archimedean method) Find the volume of n inscribed and n circumscribed square prisms of equal thickness to show that the volume of a square based pyramid is 1/3 the base area × height.

3. (Originally Eudoxos, reported in Euclid XII.10, but there, much harder than Archimedean method) Find the volume of n inscribed and n circumscribed cylindrical discs of equal thickness to show that the volume of a circular cone is 1/3 the base area × height.

4. (Fermat, 1636) Find the area of n inscribed and n circumscribed rectangles of equal width to show that the area bounded by the 'higher parabola' $y = x^3$, the x-axis and $x = a$ is $\frac{1}{4}a^4$. For this you must use $\Sigma r^3 = [\frac{1}{2}n(n + 1)]^2$ instead of $\Sigma r^2 = (n/6)(n + 1)(2n + 1)$.

5. (Fermat, 1636) Find the area of n inscribed and n circumscribed equiangular sectors to show that the area bounded by the spiral $r^2 = a\theta$ and $\theta = 0$ is one half the area of its circumscribing circle. For this you must use $\Sigma r = \frac{1}{2}n(n + 1)$.

6. (Archimedes) Find the volume of n inscribed and n circumscribed cylindrical discs of equal thickness to show that the volume of the paraboloid formed by revolving $y = x^2$ about the y-axis and cut off by the plane $y = R^2$ is one and a half times the volume of the cone with the same base and vertex O.
For this you must use $\Sigma r = \frac{1}{2}n(n + 1)$.

NEXT STEP: first use of the limit $a + ar + ar^2 + \ldots = a/(1 - r)$ for $0 < r < 1$ for area. (Fermat, 1643) By finding the area of an infinite number of inscribed and circumscribed rectangles between the abscissae $x = a, ar, ar^2, ar^3, \ldots$, where $r > 1$, show that the area bounded by the curve $y = 1/x^2$, the line $x = a$ and the x-axis is $1/a$.